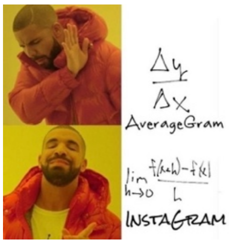
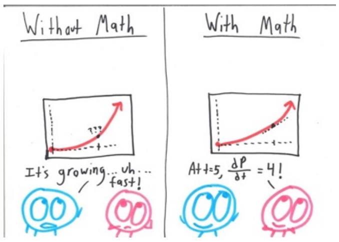
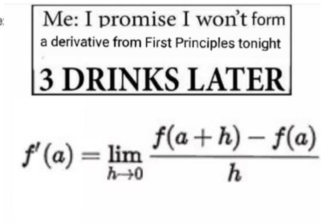
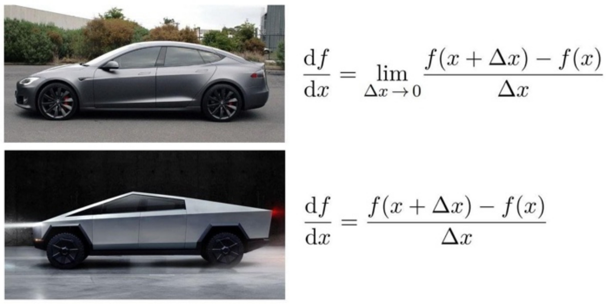
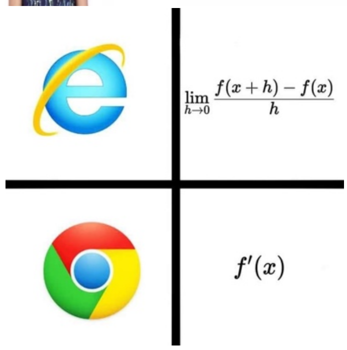
Differentiation From First Principles:











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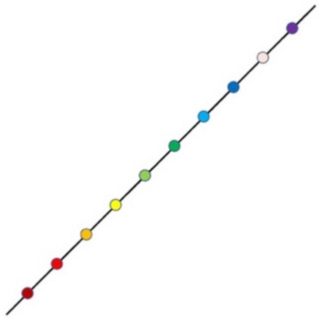
1 Background Information – Lines versus curves

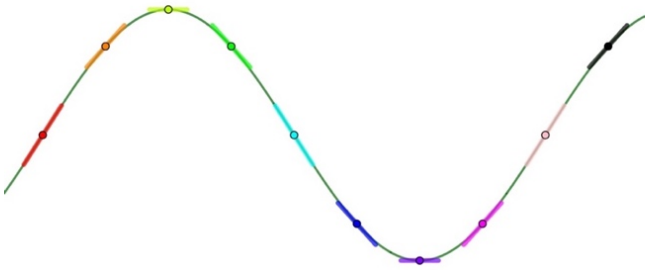
Up until we learn differentiation, we can only find the gradient of a **straight line** i.e. the gradient between two points which is $\frac{rise}{run}$. Notice how the gradient is the same at every single point on the straight line to the right. If we pick any two points (x_1, y_1) and (x_2, y_2) we get the gradient $\frac{y_2-y_1}{x_2-x_1}$.

The answer for the gradient will be the same no matter which two points we pick.

Note: Gradient is also referred to as slope or rate of change (also called average rate of change).



What about the gradient of a **curve**? You can see from the coloured lines drawn at each of the points below (these are known as **tangent lines**) that the gradient changes at every point (the tangent lines do not have the same slope at every point – some are steeper than others and some slopes are negative and some are positive).

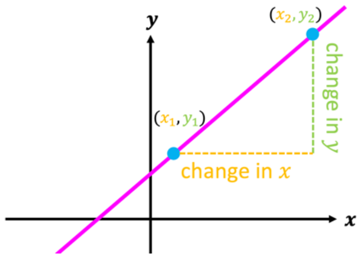


Because the gradient is now **constantly** changing at different points along the curve $f(x)$, we no longer want to find the average slope between 2 points, but instead the slope at a point (so we need to find the slope at a point not between points so we can no longer use the slope formula $\frac{y_2-y_1}{x_2-x_1}$). We can draw the coloured tangent lines as shown to approximate the gradient at different points. The slope of a curve $f(x)$ at a point means the slope of the tangent at that point i.e. the gradient of a curve at a point is defined as the slope of a tangent line at that particular point (this means finding the slope of the tangent lines at each points finds us the gradient at each of the points). To find an **approximation** of the gradient we can any pick any two points on the tangent lines drawn and calculate the gradient using your familiar formula $\frac{\text{rise}}{\text{run}}$ or $\frac{y_2-y_1}{x_2-x_1}$. Using the tangent lines in this way only gives <u>an approximation</u> though! We want <u>the exact gradient</u> at these points which differentiation allows us to find.

2 Differentiation Introduction

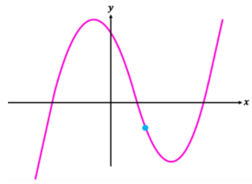
Finding the slope **between two points** above is known as finding an **average rate of change** (left diagram below), whereas finding a slope evaluated **at a single point** (via differentiation) is called the **instantaneous rate of change** (right diagram below).

<table>
<tr><th colspan="2">Rates of Change</th></tr>
<tr><th><u>Average</u> rate of change – slope <u>between</u> 2 points</th><th><u>Instantaneous</u> rate of change – slope <u>at</u> a point</th></tr>
<tr><td>
Pick any 2 points on the line



$$\text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{y_1 - y_2}{x_1 - x_2}$$

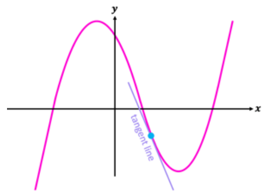
Note: It should make sense that this is called average because we finding the slope <u>between</u> points.
</td><td>
Pick any point on the curve



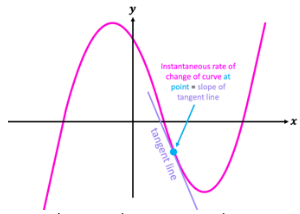
How do we find the slope at a point? There is nothing to measure now that we no longer have 2 points. There is no change in y and no change in x.

$$\text{slope} = \frac{0}{0} = ???$$

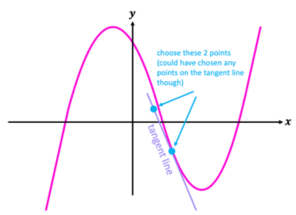
Instead, we can draw a tangent line to the curve at the point (a tangent line touches the curve at a single point and does not cross through the curve).



The tangent line will have the same slope as the curve at the point.



To find the slope so far we have always used 2 points with the formula $\frac{y_2 - y_1}{x_2 - x_1}$, right? Well, we could still do this by picking any 2 points on the tangent line and proceeding as on the left.



There is only one possible tangent line that can be drawn at a single point, but drawing this tangent line by hand and reading the exact coordinates of the points off of the graph isn't very accurate and only gives us an approximation. We need a more accurate method using 2 points ON the curve and secant line which is known as differentiation from first principles!
</td></tr>
</table>

The following examples and further explanations below are not that vital to know in order to be able to answer questions on first principles for your course. You can skip straight to the method section in chapter 3 if you wish.

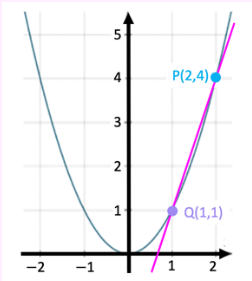
You may already be familiar with how to differentiate the short way (don't worry if you are not yet) and soon (if not already) might be wondering why we need to go though this long process of differentiation from first principles. It is something of a bruteforce method for calculating a derivative – the technique explains how the idea of differentiation first came to being. Differentiation by first principles is the long way to differentiate and you have to know how.

The idea behind differentiation first principles is by taking **two points on the curve** that lie very closely together, the straight line between them will have approximately the same gradient as the tangent line there.

As already mentioned, for a straight line we can find the average slope between 2 points, but when finding the slope at a point now there seems to be nothing to measure. With differentiation we use a small difference (2 points on the curve that lie very close together) and have it shrink towards zero. Let's look at a numerical example before introducing differentiation by first principles

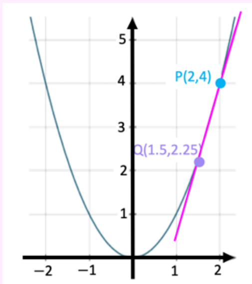
Find the slope of the curve $y = x^2$ at the point P(2,4)

Let's choose a point not too far away from P. So, let's choose the point Q$(1,1)$ which is somewhere near $P(2,4)$.



The slope of PQ is given by $\frac{4-1}{2-1} = 3$.

Now we move Q closer to P. Let's use $Q(1.5, 2.25)$ which is closer to P(2,4).



The slope of PQ is now given by $\frac{4-2.25}{2-1.5} = 3.5$. We see that this line is already a pretty good approximation to the tangent at P (since the line in the first diagram did not look like a tangent at P, but the line in the second diagram looks more like a tangent to P i.e. touches the curve once), but it is still not good enough.

So let's now we move Q even closer to P, say Q(1.9,3.61). The slope of PQ is now given by $\frac{4-3.61}{2-1.9} = 3.9$

Now if Q is moved to (1.99,3.9601), then slope PQ is 3.99

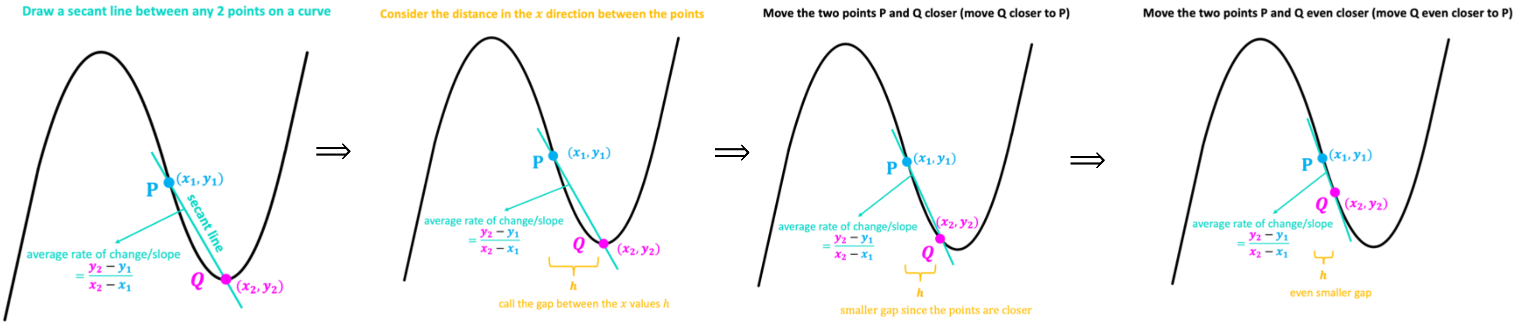
If Q is moved to (1.999, 3.996001), then the slope is 3.999

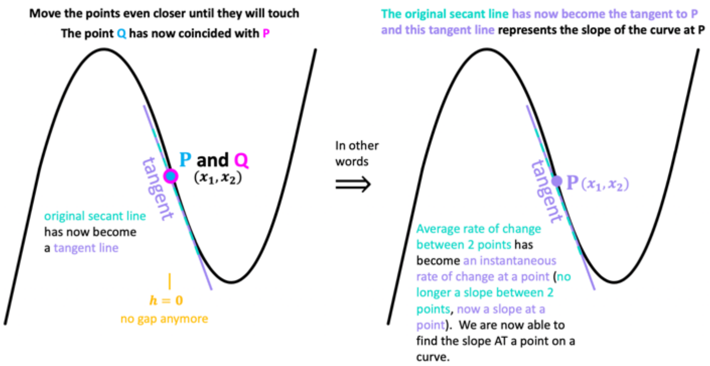
Clearly, as $x \to 2$, , the slope of PQ → 4. But notice that we cannot actually let $x = 2$ since the fraction for the slope would have 0 on the bottom and would be undefined.

We have found that the slope i.e. rate of change of y with respect to x at the point is 4 units at the point $x = 2$

First principles works like this example. We want to work out the slope at a point on a curve and to do this we choose another point (since we need 2 points to find the gradient) and keep moving that point closer (close the gap between them) until there is no gap and hence the point chosen coincides with the point we wanted to find the gradient at. Hence, we no longer have the derivative between two points (average rate of change), but the derivative at a single point P (instantaneous rate of change).

Let's first draw a picture of the reasoning behind first principles before introducing the first principles formula

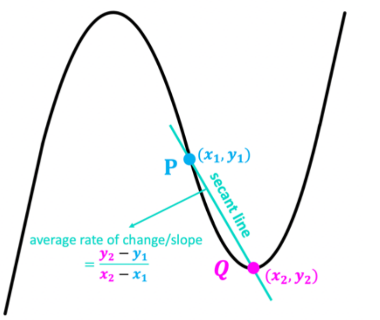




In words these diagrams say: we close the gap h between the two points P and Q (by letting **h tend to zero)** and this distance h becomes so small that the poin becomes P and hence we no longer have the derivative between two points (average rate of change), but the derivative at a single point P (instantaneous rate change). The secant line to the curve instead becomes the tangent line to the curve which is what differentiation is all about, **finding the gradient of a tangent drawn at any point to the curve**. The gradient of the tangent is the same as the gradient of the curve at the point at which it is drawn. If we let Q go all the wa to touch P such as in the final diagram (i.e. h=0) then we would have the **exact** slope of the tangent. The original purple secant has now coincided with the turquoise tangent line i.e. become the tangent. Remember a tangent line by definition is just a straight line that touches the curve at a single point.

Let's write this example generally and introduce the first principles formula $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ which looks a bit scary, but isn't once you understand the reasonin above.

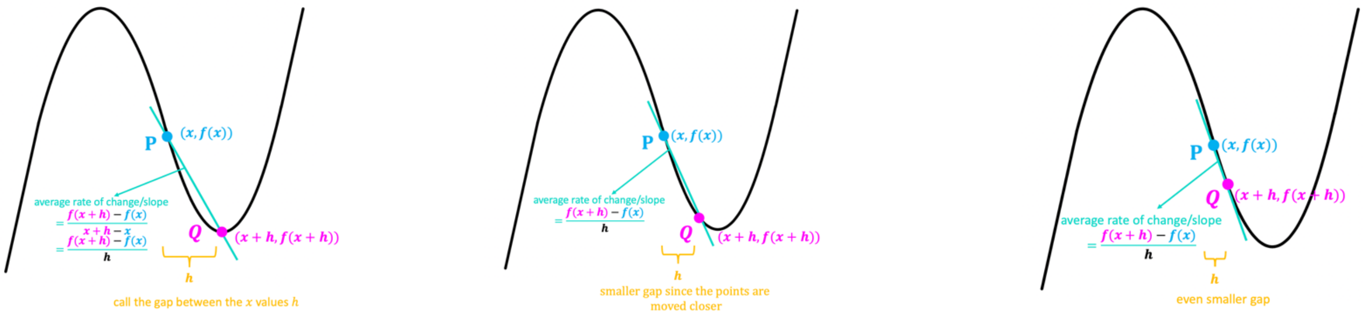
Draw a secant line between any 2 points on a curve



To find the derivative of the line PQ we use the slope formula $\frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

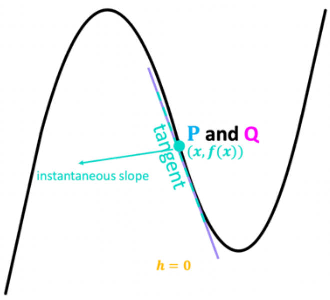
If we move Q closer and closer to P (that is, we let h get smaller and smaller), the line PQ will get closer and closer to being the tangent at P and so the slope of PQ gets closer to the red slope that we want (remember the slope of the tangent at the point is the slope of the curve at the point)

Instead of using the usual points (x_1, y_1) and (x_2, y_2), we use $(x, f(x))$ and $(x+h, f(x+h))$. Remember that writing a function notation $f(x)$ the same as saying y so this should make sense that the $y's$ change to $f(x)'s$. x changes from x to $x+h$ and y changes from $f(x)$ to $f(x+h)$. Why did we introduce h? It allows us a to be able to talk about moving the point Q closer to P as we will see below. The gradient of line PQ is $\frac{change\ in\ y}{change\ in\ x} = \frac{\mathrm{y_2 - y_1}}{\mathrm{x_2 - x_1}} = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{x+h-x}$. Simplified this becomes $\frac{f(x+h)-f(x)}{h}$
Note: Some teachers will use Δx instead of h so you will see $\frac{f(\Delta x + h) - f(\Delta x)}{\Delta x}$. x changes from x to $x + \Delta x$ and y changes from $f(x)$ to $f(x+\Delta x)$. Δx means change in x.
$\frac{f(x+h)-f(x)}{h}$ is almost the formula for first principles. We need a limit at the front which is missing. Consider the following scenario where we close the gap between the points.



So, by letting **h tend to zero** i.e. $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$ we close the gap h between the two points P and Q and this distance h becomes so small that the point Q becomes P and hence we no longer have the derivative between two points (average rate of change), but the derivative at a single point P (instantaneous rate of change). The secant line to the curve instead becomes the tangent line to the curve which is what differentiation is, **finding the gradient of a tangent drawn at any point to the curve**. The gradient of the tangent is the same as the gradient of the curve at the point at which it is drawn.

If we let Q go all the way to touch P such as in the diagram on the right (i.e. h=0) then we would have the **exact** slope of the tangent. The original green secant line has now coincided with the purple tangent line i.e. become the tangent. Remember a tangent line by definition is just a straight line that touches the curve at a single point.

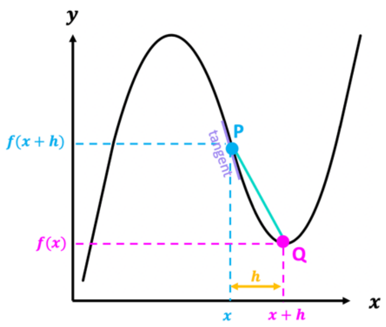


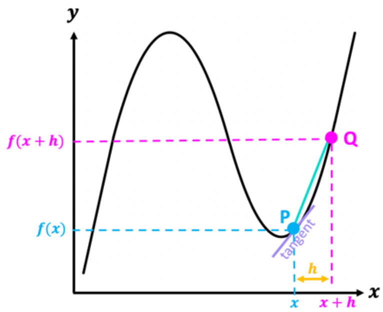
This formula $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$ is called the formal definition of the derivative is known as differentiation by first principles.

So, in summary this looks like

Negative Gradient

Positive gradient





And we have

$$\lim_{h\to 0}(\text{gradient of the line}) = \lim_{h\to 0}\frac{y_2-y_1}{x_2-x_1} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{x+h-x} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Extension:

Note: We didn't have to use the points $(x, f(x))$ and $(x+h, f(x+h))$

We could have used a distance h to the left of the point $(x, f(x))$ (instead of to the right) and hence used the points $(x-h, f(x-h))$ and $(x, f(x)$. This means we could also see the formula for first principles written as $\lim_{h\to 0}\frac{f(x)-f(x-h)}{h}$

We could have also used the points $(x+h, f(x+h))$ and $(x-h, f(x-h))$. This means we could also see the formula for first principles written as $\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$

3 Step by Step Method And Examples

3.1 Method

For this topic we need to find $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. This formula looks a bit scary and daunting, but if we take it step by step it is easy.

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Plugging any equation into this formula will result in the spitting out of our derivative function.

Note: Some formula use Δx instead of h. This notation just unnecessarily overcomplicates things. Using h is better.

We find each piece of the formula in stages.

Step 1: Re-write the given function y in the question as $f(x)$. For example, if given $y = x^2 - 2$ then we write $f(x) = x^2 - 2$

Step 2: Find $f(x+h)$ using function knowledge i.e. replace every x in the function with $x + h$

Step 3: Fill $f(x)$ and $f(x+h)$ from steps 1 and 2 into the numerator of $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Step 4: Simplify the numerator of $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ i.e. simplify $f(x+h) - f(x)$.

- If polynomials: expand brackets (use binomial expansion if necessary i.e. if the power of the bracket is higher than 2 or 3 and
  then collect like terms)
- If fractions: get a common denominator and subtract the fractions
- If roots: rationalise

Step 5: Find $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

We now find the limit by replacing h with 0 in our function from step 4

This means we find the limit of $\frac{f(x+h)-f(x)}{h}$ by replacing every h with 0 in the numerator and denominator.

Once we plug $h = 0$ in, the $\lim_{h \to 0}$ notation is no longer written.

(If you end up with an answer of $\frac{0}{0}$ from doing this you must go back and factorise first and cancel or rationalise and cancel and

then plug $h = 0$ in again. Now you should not get $\frac{0}{0}$). Watch out for trig types though as they work a bit differently as you need to

use trig identities to simplify in step 4 – see examples 7 and 8.

3.2 Examples

Seeing worked examples are the best way to understand

3.2.1 Polynomials

Example 1:

Differentiate $y = 3x^2$ from first principles

Formula: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Don't worry about the lim part and the denominator h until the end

Step 1: We write $y = 3x^2$ as $f(x) = 3x^2$ (Note: $f(x)$ is always our function given in the question)

Step 2: We find $f(x+h)$ which means we replace every x in $3x^2$ with $x+h$

$$f(x+h) = 3(x+h)^2$$

Step 3: We plug what we have found into steps 1 and 2 into $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\lim_{h\to 0}\frac{3(x+h)^2 - 3x^2}{h}$$

Step 4: Simplify the numerator (expand the brackets and collect like terms)
Careful with the first bracket in the numerator. We expand $(x+h)^2$ and then multiply by 3

$$\lim_{h\to 0}\frac{3(x^2+2xh+h^2)-3x^2}{h} = \lim_{h\to 0}\frac{3x^2+6xh+3h^2-3x^2}{h} = \lim_{h\to 0}\frac{6xh+3h^2}{h}$$

Step 5: Substitute $h = 0$ into $\lim_{h\to 0}\frac{6xh+3h^2}{h}$ produces

$$\frac{6x(0)+3(0)^2}{0} = \frac{0}{0}$$

Getting $\frac{0}{0}$ is an indeterminate form which tells us nothing
About the limit. To find the limit you can factorise h out in the numerator and then cancel out the common factors h. The h from the numerator cancels out with the h in the denominator

$$\lim_{h\to 0}\frac{6xh+3h^2}{h} = \lim_{h\to 0}\frac{\cancel{h}(6x+h)}{\cancel{h}} = \lim_{h\to 0}(6x+h)$$

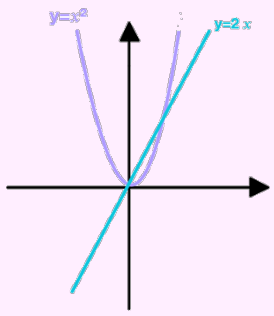
Now we replace h with 0 again

$$6x + 0 = 6x$$

Comments:

If you already know how to differentiate (the quick way) you can check your answer. First principles is just the long way. We know $3x^2$ becomes $6x$ when we differentiate (the quick rule is to bring the power down and take one off of the power) and hence we are correct!

The graphs of the function x^2 and its derivative $2x$ look like:



Notice how the graph has one less turn. It goes from a quadratic to a straight line

Example 2:

Prove from first principles that the derivative of 8 is zero

Formula: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Step 1: We write $y = 8$ as $f(x) = 8$

Step 2: We find $f(x+h)$ which means we replace every x in x^2 with $x+h$. There are no $x's$ so it stays the same.

$$f(x+h) = 8$$

Step 3: We plug what we have found into steps 1 and 2 into $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\lim_{h\to 0} \frac{8-8}{h}$$

Step 4: Simplify the numerator

$$\lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 \text{ since } \frac{0}{\text{anything}} \text{ is always } 0$$

Step 5: Substitute $h = 0$ into $\lim\limits_{h\to 0} 0$. There is no h here so it is always 0 regardless of the value of h.

Example 3:

Differentiate $y = 3x^3 - x - 9$ using first principles

Formula: $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$f(x) = 3x^3 - x - 9$$

$$f(x+h) = 3(x+h)^3 - (x+h) - 9$$

$\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$ becomes $\lim\limits_{h\to 0} \frac{3(x+h)^3-(x+h)-9-(3x^3-x-9)}{h}$

Careful with the – sign. We have to multiply all the terms in the second bracket by the negative.

Careful with the first bracket in the numerator. We do $(x+h)^3$ for the first bracket and then multiply by 3

$$\lim_{h\to 0} \frac{3(x^3+3x^2h+3xh^2+h^3) - x - h - 9 - 3x^3 + x + 9}{h}$$

We simplify the numerator

$$= \lim_{h\to 0} \frac{3x^3+9x^2h+9xh^2+3h^3 - x - h - 9 - 3x^3 + x + 9}{h}$$

$$\lim_{h\to 0} \frac{9x^2h+9xh^2+3h^3-h}{h}$$

Factorise h out in the numerator and then cancel out the common factors h.

$$\lim_{h\to 0} \frac{h(9x^2+9xh+3h^2-1)}{h}$$

$$\lim_{h\to 0} 9x^2+9xh+3h^2-1$$

We find the limit by replacing h with 0

$$\lim_{h\to 0} 9x^2+9x(0)+3(0)^2-1$$

$$= 9x^2 - 1$$

3.2.2 Fractions

These questions can often have fractions or roots (or combinations of both) which are a bit harder in terms of the algebra required for the simplification.

Example 4:

Show that the derivative of $\frac{4}{x}$ is $-\frac{4}{x^2}$ using first principles

We don't need to show all the steps now since you should understand how the process works, but the simplification of the fractions will be shown in pink.

$$f(x) = \frac{4}{x}$$
$$f(x+h) = \frac{4}{x+h}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\lim_{h\to 0}\frac{\frac{4}{x+h}-\frac{4}{x}}{h}$$

Get common denominator in numerator

$$\lim_{h\to 0}\frac{\frac{4x-4(x+h)}{x(x+h)}}{h}$$

$$\lim_{h\to 0}\frac{\frac{-4h}{x(x+h)}}{h}$$

Cancel h's

$$\lim_{h\to 0}\frac{-4}{x(x+h)}$$

We find the limit by replacing h with 0

$$\frac{-4}{x^2}$$

3.2.3 Roots

Example 5:

Show that the derivative of $\sqrt{3x-1}$ is $\frac{3}{2\sqrt{3x-1}}$ using first principles

$$f(x) = \sqrt{3x-1}$$
$$f(x+h) = \sqrt{3(x+h)-1}$$

Fill into the formula $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\lim_{h\to 0}\frac{\sqrt{3(x+h)-1}-\sqrt{3x-1}}{h}$$

$$\lim_{h\to 0}\frac{\sqrt{3x+3h-1}-\sqrt{3x-1}}{h}$$

We need rationalise before taking the limit otherwise we end up with $\frac{0}{0}$ if we plug $h=0$ in above

$$\lim_{h\to 0}\frac{\sqrt{3x+3h-1}-\sqrt{3x-1}}{h}\times\frac{\sqrt{3x+3h-1}+\sqrt{3x-1}}{\sqrt{3x+3h-1}+\sqrt{3x-1}}$$

$$\lim_{h\to 0}\frac{(\sqrt{3x+3h-1}-\sqrt{3x-1})(\sqrt{3x+3h-1}+\sqrt{3x-1})}{h(\sqrt{3x+3h-1}+\sqrt{3x-1})}$$

Middle terms cancel when you expand the numerator

$$\lim_{h\to 0}\frac{(3x+3h-1)-(3x-1)}{h\left(\sqrt{3x+3h-1}+\sqrt{3x-1}\right)}$$

$$\lim_{h\to 0}\frac{3x+3h-1-3x+1}{h\left(\sqrt{3x+3h-1}+\sqrt{3x-1}\right)}$$

$$\lim_{h\to 0}\frac{3h}{h\left(\sqrt{3x+3h-1}+\sqrt{3x-1}\right)}$$

$$\lim_{h\to 0}\frac{3}{\sqrt{3x+3h-1}+\sqrt{3x-1}}$$

We find the limit by replacing h with 0

$$=\frac{3}{\sqrt{3x-1}+\sqrt{3x-1}}$$

$$=\frac{3}{2\sqrt{3x-1}}$$

Example 6:

Differentiate $\frac{1}{\sqrt{x-1}}$ using first principles

$$f(x)=\frac{1}{\sqrt{x-1}}$$

$$f(x+h)=\frac{1}{\sqrt{x+h-1}}$$

Fill into the formula $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h-1}}-\frac{1}{\sqrt{x-1}}}{h}$$

Get a common denominator in numerator (turn into one fraction)

$$\lim_{h\to 0}\frac{\frac{\sqrt{x-1}-\sqrt{x+h-1}}{\sqrt{x+h-1}\sqrt{x-1}}}{h}$$

$$\lim_{h\to 0}\frac{\sqrt{x-1}-\sqrt{x+h-1}}{h\sqrt{x+h-1}\sqrt{x-1}}$$

Rationalise

$$\lim_{h\to 0}\frac{\sqrt{x-1}-\sqrt{x+h-1}}{h\sqrt{x+h-1}\sqrt{x-1}}\times\frac{\sqrt{x-1}+\sqrt{x+h-1}}{\sqrt{x-1}+\sqrt{x+h-1}}$$

$$\lim_{h\to 0}\frac{x-1-(x+h-1)}{h\sqrt{x+h-1}\sqrt{x-1}\left(\sqrt{x-1}+\sqrt{x+h-1}\right)}$$

$$\lim_{h\to 0}\frac{x-1-(x+h-1)}{h\sqrt{x+h-1}\sqrt{x-1}\left(\sqrt{x-1}+\sqrt{x+h-1}\right)}$$

$$\lim_{h\to 0}\frac{-h}{h\left(\sqrt{x+h-1}\sqrt{x-1}\left(\sqrt{x-1}+\sqrt{x+h-1}\right)\right)}$$

$$\lim_{h\to 0}\frac{-1}{\sqrt{x+h-1}\sqrt{x-1}\left(\sqrt{x-1}+\sqrt{x+h-1}\right)}$$

We find the limit by replacing h with 0

$$\frac{-1}{(x-1)\left(2\sqrt{x-1}\right)}$$

$$\frac{-1}{2(x-1)^{\frac{3}{2}}} = -\frac{1}{2}(x-1)^{-\frac{3}{2}}$$

3.2.4 Trigonometry

Example 7:

Prove from first principles that the derivative of $\sin\theta$ is $\cos\theta$ using the results that as $h \to 0, \frac{\sin h}{h} \to 1$ *and* $\frac{\cos h - 1}{h} \to 0$

$$f(\theta) = \sin\theta \text{ and } f(\theta+h) = \sin(\theta+h)$$

Fill into the formula $\lim_{h\to 0} \frac{f(\theta+h)-f(\theta)}{h}$

$$\lim_{h\to 0} \frac{\sin(\theta+h) - \sin\theta}{h}$$

Use the addition/compound angle formula

$$\lim_{h\to 0} \frac{\sin\theta\cos h + \cos\theta\sin h - \sin\theta}{h}$$

Factorise out the common term $\sin\theta$ from the first and last term

$$\lim_{h\to 0} \frac{\sin\theta(\cos h - 1) + \cos\theta\sin h}{h}$$

Take out everything without h in it (since the limit only depends on h so terms without an h don't need to be inside)

and split the limit up into 2 separate limits

$$\sin\theta \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos\theta \lim_{h\to 0} \frac{\sin h}{h}$$

We use the results $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

$$\sin\theta(0) + \cos\theta\ (1)$$

$$= \cos\theta$$

Example 8:

Prove from first principles that the derivative of $\sin\theta$ is $\cos\theta$ using small angle approximations

$$f(\theta) = \sin\theta \text{ and } f(\theta+h) = \sin(\theta+h)$$

Fill into the formula $\lim_{h\to 0} \frac{f(\theta+h)-f(\theta)}{h}$

Use the addition/compound angle formula

$$f'(\theta) = \lim_{h\to 0} \frac{\sin(\theta+h) - \sin\theta}{h}$$

$$f'(\theta) = \lim_{h\to 0} \frac{\sin\theta\cos h + \cos\theta\sin h - \sin\theta}{h}$$

Factorise out the common term $\sin\theta$ from the first and last term

$$f'(\theta) = \lim_{h\to 0} \frac{\sin\theta(\cos h - 1) + \cos\theta \sin h}{h}$$

Take out everything without h in it (since the limit only depends on h so terms without an h don't need to be inside) and split the limit up into 2 separate limits

$$f'(\theta) = \sin\theta \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos\theta \lim_{h\to 0} \frac{\sin h}{h}$$

Noe use the small angle results

$$f'(\theta) = \sin\theta \lim_{h\to 0} \frac{\left(1-\frac{h^2}{2}\right)-1}{h} + \cos\theta \lim_{h\to 0} \frac{h}{h}$$

Simplify the numerator of t he first fraction

$$f'(\theta) = \sin\theta \lim_{h\to 0} \left(\frac{-\frac{h^2}{2}}{h}\right) + \cos\theta \lim_{h\to 0} 1$$

Simplify the first fraction again

$$f'(\theta) = \sin\theta \lim_{h\to 0} \left(-\frac{h}{2}\right) + \cos\theta \lim_{h\to 0} 1$$

Let h tend to 0 (replace h with 0)

$$f'(\theta) = \sin\theta\left(-\frac{0}{2}\right) + \cos\theta\ (1)$$

$$f'(\theta) = \cos\theta$$

4 At A Point

If we want to find the actual value of the slope and not just the slope in terms of x we can either

- **Way 1:** Replace the x value in the formula $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ and then proceed as normal with first principles

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Note: h still tends to/approaches 0. This part does not change.

- **Way 2:** Use first principles first to find your answer as normal and then plug the value of x in after

Find the derivative of $y=3x^2$ at the point $x=4$ from first principles

Way 1:	Way 2:
Formula: $\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}$	Formula: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
$\lim_{h\to 0}\frac{3(4+h)^2-3(4)^2}{h}$	$\lim_{h\to 0}\frac{3(x+h)^2-3x^2}{h}$
$\lim_{h\to 0}\frac{3(4^2+8h+h^2)-48}{h}$	$\lim_{h\to 0}\frac{3(x^2+2xh+h^2)-3x^2}{h}$
$=\lim_{h\to 0}\frac{48+24h+3h^2-48}{h}$	$=\lim_{h\to 0}\frac{3x^2+6xh+3h^2-3x^2}{h}$
$=\lim_{h\to 0}\frac{24h+3h^2}{h}$	$=\lim_{h\to 0}\frac{6xh+3h^2}{h}$
$=24$	$=\lim_{h\to 0}(6x+h)$
	$=6x$
	Now we plug in $x=4$
	$6(4)=24$

5 Alternative Formulae

There is more than one form of the definition of the derivative and some courses may use an alternate formula

- **Generally (at <u>any</u> point x):**

  Recall that we have the formula $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

  Some courses may use the alternate form $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$

  The alternate form should make sense. Since the derivative is the limiting slope of the secant line, we can think as the first point at x_0 be fixed and then second point be a varying value x that tends towards x_0. By letting x tend to x_0 we close the gap between the points.

Find the derivative of $y = 3x^2$ from first principles

Alternate formula $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$

$$\lim_{x\to x_0}\frac{3x^2-3{x_0}^2}{x-x_0}$$

We can't replace x with x_0 otherwise we would get $\frac{0}{0}$ which tells us nothing. Instead we factorise first.

$$\lim_{x\to x_0}\frac{3(x^2-{x_0}^2)}{x-x_0}$$

$$\lim_{x\to x_0}\frac{3(x+x_0)(x-x_0)}{x-x_0}$$

$$\lim_{x\to x_0} 3(x+x_0)$$

Now we can replace x with x_0

$$= 3(x_0+x_0)$$

$$= 3(2x_0)$$

$$= 6x_0$$

Derivative is $6x$

- **Specifically (at a point $x = a$):**

  Recall that we have the formula $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

  Some courses may use the alternate form $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

Find the derivative of $y = 3x^2$ at the point $x = 4$ from first principles

Alternate formula $\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}$

$$\lim_{x\to 4}\frac{3x^2-3(4)^2}{x-4}$$

$$\lim_{x\to 4}\frac{3x^2-48}{x-4}$$

We can't replace x with 4 otherwise we would get $\frac{0}{0}$ which tells us nothing. Instead, we factorise first.

$$\lim_{x\to 4}\frac{3(x^2-16)}{x-4}$$

$$\lim_{x\to 4}\frac{3(x+4)(x-4)}{x-4}$$

$$\lim_{x\to 4}3(x+4)$$

Now we can replace x with 4

$$= 3(4+4) = 24$$

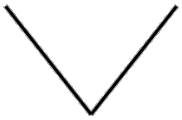
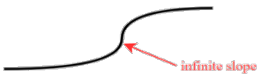
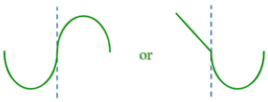
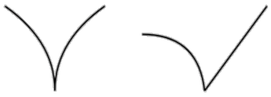
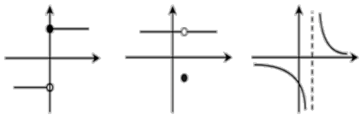
- **To show a derivative exists:**

  You need to have studied one sided limits to understand this. The left hand limit equals right hand limit $\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h}$

  Some courses may use the alternate form: $\lim_{x\to {x_0}^+}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to {x_0}^-}\frac{f(x)-f(x_0)}{x-x_0}$

6 Using First Principles to Show When A Derivative Does and Doesn't Exist

A derivative doesn't exist for one of 4 reasons. Let's take a look at what this looks like visually:

Corner/Node (sharp turn)	**Vertical Tangent**	**Cusps "concave corners"**	**Discontinuities**
Infinitely many tangent lines $y = \|x\|$ 	An infinite slope $y = x^{\frac{1}{3}}$  **or** 	sharp corners $y = x^{\frac{2}{3}}$ 	Jump Removable Infinite/Break  $f(x) = \begin{cases} 1, x \geq 0 \\ -1, x < 0 \end{cases}$ Remember differentiability ≠continuity, so if continuous, can't be differentiable

How do we show a limit doesn't exist formally? We use the definition of a derivative i.e. first principles!

Formally, if a derivative exists, the limit $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ must exist and left-hand limit must equal right hand limit.

Let's look at a few examples.

Example 1

$$f(x) = \begin{cases} x^3+1, x<1 \\ 3x-1, x \geq 1 \end{cases}$$

Determine whether the derivative exists at $x = 1$

Hint: where we usually write x in our definition of the derivative we now write 1

Way 1: Using form 1 $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Left limit:

$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+}\frac{((1+h)^3+1)-2}{h} = \lim_{h\to 0^+}\frac{3h+3h^2+h^3}{h}$
$= \lim_{h\to 0^+}\frac{h(3+3h+h^2)}{h} = \lim_{h\to 0^-}(3+3h+h^2) = 3+0+0 = 3$

Right limit:

$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-}\frac{3(1+h)-1-2}{h} = \lim_{h\to 0^-}\frac{3(1+h)-3}{h} = \lim_{h\to 0^-}\frac{3h}{h} =$
$\lim_{h\to 0} 3 = 3$

So f is differentiable at 1 since both derivatives (left and right) exist and are equal

Way 2: Using form 2 $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$

Left limit:

$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-}\frac{(x^3+1)-2}{x-1} = \lim_{x\to 1^-}\frac{x^3-1}{x-1} = \lim_{x\to 1^-}\frac{(x-1)(x^2+x+1)}{x-1}$
$= \lim_{x\to 1^-}(x^2+x+1) = 1+1+1 = 3$

Right limit:

$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+}\frac{(3x-1)-2}{x-1} = \lim_{x\to 1^+}\frac{3x-3}{x-1} = \lim_{x\to 1^+}\frac{3(x-1)}{x-1} = \lim_{x\to 1^+} 3 = 3$

So f is differentiable at 1 since both derivatives (left and right) exist and are equal

Example 2

$$f(x) = x^{\frac{1}{3}}$$

Determine whether the derivative exists at $x = 0$

Hint: where we usually write x in our definition of the derivative we now write 0

Way 1: Using form 1 $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+}\frac{((0+h)^{\frac{1}{3}})-0}{h} = \lim_{h\to 0^+}\frac{h^{\frac{1}{3}}}{h} = \lim_{h\to 0^+}\frac{1}{h^{\frac{2}{3}}} = \infty$

$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-}\frac{((0+h)^{\frac{1}{3}})-0}{h} = \lim_{h\to 0^-}\frac{h^{\frac{1}{3}}}{h} = \lim_{h\to 0^-}\frac{1}{h^{\frac{2}{3}}} = -\infty$

Way 2: Using form 2 $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$

$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{x^{\frac{1}{3}}-0}{x} = \lim_{x\to 0^+} x^{-\frac{2}{3}} = \lim_{x\to 0^+}\frac{1}{x^{\frac{2}{3}}} = \frac{1}{0^+} = \infty$

$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{x^{\frac{1}{3}}-0}{x}\ \lim_{x\to 0^-} x^{-\frac{2}{3}} = \lim_{x\to 0^-}\frac{1}{x^{\frac{2}{3}}} = \frac{1}{0^-} = -\infty$

So f is not differentiable at 0 since both derivatives (left and right) are not equal (and don't even exist)	So f is not differentiable at 0 since both derivatives (left and right) are not equal and don't even exist)